MATH 512, FALL 14 COMBINATORIAL SET THEORY WEEK 1

Let κ be a regular (i.e. $\mathrm{cf}(\kappa) = \kappa$) uncountable cardinal.

Definition 1. *A set $C \subset \kappa$ is closed and unbounded (club) in κ if*

- *for all $\alpha < \kappa$, there is $\beta \in C \setminus \alpha$, and*
- *for all increasing sequences $\langle \alpha_i \mid i < \tau \rangle$ of ordinals in C for some $\tau < \kappa$, $\sup_{i<\tau} \alpha_i \in C$.*

Some examples: $\kappa \setminus \alpha$, for all $\alpha < \kappa$; the set of limit ordinals less than κ.

Lemma 2. *Suppose that C, D are clubs in κ. Then so is $C \cap D$.*

Proof. To show closure, suppose that $\tau < \kappa$ and $\langle \alpha_i \mid i < \tau \rangle$ is an increasing sequence of ordinals in $C \cap D$, and let $\alpha = \sup_{\xi<\tau} \alpha_\xi$. Since C is closed, we have that $\alpha \in C$. Since D is closed, we have that $\alpha \in D$.

To show unboundedness, fix $\alpha < \kappa$. Let $\alpha_0 > \alpha$ be a point in C. Then let $\beta_0 > \alpha_0$ be a point in D. Continue inductively, to build increasing sequences $\langle \alpha_n \mid n < \omega \rangle$ of points in C and $\langle \beta_n \mid n < \omega \rangle$ of points in D, such that for all n, $\alpha_n < \beta_n < \alpha_{n+1}$. Then $\beta := \sup_n \alpha_n = \sup_n \beta_n \in C \cap D$, and $\beta > \alpha$. $\square$

Definition 3. *A set $S \subset \kappa$ is stationary if for all clubs C, $S \cap C \neq \emptyset$*

Some examples: every club set, $E^\kappa_\omega := \{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \omega\}$. Also note that if S is stationary and C is a club, then $S \cap C$ is stationary.

Definition 4. *Let $\langle A_\xi \mid \xi < \kappa \rangle$ be subsets of κ. The diagonal intersection is defined to be $\triangle_{\xi<\kappa} A_\xi := \{\beta < \kappa \mid \beta \in \bigcap_{\xi<\beta} A_\xi\}$. A filter is called* **normal** *if it is closed under diagonal intersections.*

Proposition 5.

1. *If $\langle C_\xi \mid \xi < \tau \rangle$ are clubs in κ for some $\tau < \kappa$, then $\bigcap_{\xi<\tau} C_\xi$ is also a club.*
2. *If $\langle C_\xi \mid \xi < \kappa \rangle$ are clubs in κ, then $\triangle_{\xi<\tau} C_\xi$ is a club.*

The **club filter** on κ is the collection of all subsets of κ containing a club. If follows by the above that the club filter is a normal κ-complete filter on κ, and it contains all complements of bounded sets.

Theorem 6. *(Fodor) Suppose that $S \subset \kappa$ is stationary and $f : S \to \kappa$ is a* **regressive** *function, i.e. $f(\alpha) < \alpha$ for all $\alpha \in S$. Then there is a stationary $T \subset S$, such that f is constant on T.*

Proof. Otherwise, for all $\gamma < \kappa$, $f^{-1}(\gamma)$ is nonstationary, i.e. there is a club C_γ with $C_\gamma \cap f^{-1}(\gamma) = \emptyset$. Let $C = \triangle_{\gamma<\kappa} C_\gamma$, and let $\alpha \in C \cap S$. Set

$\gamma := f(\alpha)$. Since f is regressive, $\gamma < \alpha$, and so by the definition of diagonal intersection, $\alpha \in C_\gamma$. But $\alpha \in f^{-1}(\gamma)$. Contradiction.

□

The conclusion of this theorem is actually a necessary and sufficient condition for normality. An application of Fodor's theorem is the following fact:

Fact (Solovay): Every stationary subset S of κ can be partitioned into κ many disjoint stationary subsets. (for the proof, see Chapter 8 of Jech)

Definition 7. *A cardinal κ is inaccessible if it is regular and strong limit (i.e. $\tau < \kappa \to 2^\tau < \kappa$).*

Proposition 8. *Suppose κ is inaccessible. Then the set of cardinals below κ is club.*

Definition 9. *An inaccessible cardinal κ is Mahlo if the set of regular cardinals below κ is stationary.*

So far we have defined clubs and stationary subsets of a cardinal. For an ordinal α, $c \subset \alpha$ is a club in α is it is closed and unbounded in α, and $s \subset \alpha$ is stationary in α is it meets every club in α.

For a set $B \subset \beta$, $\lim(B)$ will denote the limit points of B, i.e. $\lim(B) := \{\alpha < \beta \mid B \cap \alpha \text{ is unbounded}\}$. Note that if B is unbounded, then $\lim(B)$ is a club. Also, for any club C, $\lim(C) \subset C$.

Definition 10. *Let $S \subset \kappa$ be stationary. S* **reflects** *if for some $\alpha < \kappa$, $S \cap \alpha$ is stationary in α.*

Definition 11. *For a stationary set T, $Refl(T)$ denotes the statement that every stationary subset of T reflects.*

For example, for any uncountable regular κ, E^κ_ω reflects.

Proposition 12. *Let κ be any cardinal (possibly singular), and let T be a stationary subset of $E^{\kappa^+}_\kappa$. Then T does not reflect.*

Proof. Let $\alpha < \kappa^+$ be any point. Let $C \subset \alpha$ be club in α with $o.t.(C) = \mathrm{cf}(\alpha) \leq \kappa$. Then $\lim(C)$ is a club subset of α, disjoint from T. □

Definition 13. *$\Box_\kappa$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha < \kappa^+\rangle$, such that for every α,*

- *C_α is club in α with $o.t.(C_\alpha) \leq \kappa$;*
- *if $\beta \in \lim(C_\alpha)$, then $C_\alpha \cap \beta = C_\beta$.*

Lemma 14. *$\Box_\kappa$ implies $\neg Refl(S)$ for every stationary $S \subset \kappa^+$.*

Proof. Suppose that $\langle C_\alpha \mid \alpha < \kappa^+\rangle$ is a square sequence and S is a stationary subset of κ^+. Let $F(\alpha) := o.t.(C_\alpha)$. By Fodor, there is a stationary subset $T \subset S$, such that F is constant on T. I.e. for some δ, for all $\alpha \in T$, $o.t.(C_\alpha) = \delta$. We claim that T does not reflect. For otherwise, if $T \cap \alpha$ is stationary in α, then $T \cap \lim(C_\alpha)$ is also stationary in α. Let $\beta < \gamma$ be two points in $T \cap \lim(C_\alpha)$. Then since $C_\alpha \cap \gamma = C_\gamma$, we have that $\beta \in \lim(C_\gamma)$, and so $C_\gamma \cap \beta = C_\beta$. It follows that $F(\beta) = o.t.(C_\beta) < o.t.(C_\gamma) = F(\gamma)$. Contradiction with $\beta, \gamma \in T$. $\square$

Next we give some weakenings of square:

Definition 15. *$\square_{\kappa,\lambda}$ asserts the existence of a sequence $\langle \mathcal{C}_\alpha \mid \alpha < \kappa^+ \rangle$, such that for every α,*

- *$1 \leq |\mathcal{C}_\alpha| \leq \lambda$,*
- *for every α, for every $C \in \mathcal{C}_\alpha$, C is club in α with $o.t.(C) \leq \kappa$;*
- *for every α, for every $C \in \mathcal{C}_\alpha$, for every $\beta \in \lim(C)$, we have $C \cap \beta \in \mathcal{C}_\beta$.*

The principle **weak square** is $\square^*_\kappa := \square_{\kappa,\kappa}$. We have that for any $\lambda < \kappa$, $\square_\kappa \to \square_{\kappa,\lambda} \to \square^*_\kappa$.

Lemma 16. *Suppose that $\kappa^{<\kappa} = \kappa$. Then $\square^*_\kappa$ holds.*

Proof. For every limit $\alpha < \kappa^+$ with $\mathrm{cf}(\alpha) < \kappa$, let $\mathcal{C}_\alpha := \{C \subset \alpha \mid C \text{ is a club}, |C| < \kappa\}$, i.e. all club subsets of α of size less than κ. Since $\kappa^{<\kappa} = \kappa$, we have that $|\mathcal{C}_\alpha| = \kappa$.

If κ is regular, for every limit $\alpha < \kappa^+$ with $\mathrm{cf}(\alpha) = \kappa$, let C_α be any club in α of order type κ. Set $\mathcal{C}_\alpha = \{C_\alpha\}$.

Suppose that $C \in \mathcal{C}_\alpha$ and $\beta \in \lim(C)$. Then since $|C| \leq \kappa$, we have that $\mathrm{cf}(\beta) < \kappa$ and $C \cap \beta$ is a club subset of β of size less than κ. So, by definition, $C \cap \beta \in \mathcal{C}_\beta$.

$\square$

Note that the above implies that weak square holds for all inaccessible κ. Also, under GCH, weak square will hold for every regular κ. So, we will be most interested in $\square^*_\kappa$ when κ is singular.

In general square principles are a "incompactness" type properties: a property that a structure lacks, but all of its substructures of smaller cardinality have. This is examplified in the following lemma:

Lemma 17. *Suppose that $\langle \mathcal{C}_\alpha \mid \alpha < \kappa^+ \rangle$ is a $\square_{\kappa,\lambda}$ sequence, for some $1 \leq \lambda \leq \kappa$. Then there is no club $C \subset \kappa^+$, such that for all $\alpha \in \lim(C)$, $C \cap \alpha \in \mathcal{C}_\alpha$.*

Proof. If C is a club in κ^+, let $\alpha \in \lim(C)$ be such that $o.t.(C \cap \alpha) > \kappa$. We can always find such an α, since $o.t.(C) = \kappa^+$. But for every $E \in \mathcal{C}_\alpha$, $o.t.(E) \leq \kappa$, so $C \cap \alpha \notin \mathcal{C}_\alpha$.

$\square$

Definition 18. *A* **tree** *$(T, <)$ is a partially ordered set, such that for every $x \in T$, the set of predecessors of x, $pred(x) := \{y \in T \mid y < x\}$ is well ordered by $<$. We set:*

- *for $x \in T$, $level(x) := o.t.(pred(x))$,*

- *the height of the tree,* $ht(T) := \sup\{level(x) + 1 \mid x \in T\}$*,*
- *for* $\alpha < ht(T)$*, the* α*-th level of* T *is* $T_\alpha := \{x \in T \mid level(x) = \alpha\}$*.*

We also say that $b \subset T$ *is a* **branch***, if* b *is a maximal linearly ordered subset of* T*.*

Note that if b is a branch, then for every level α, $|b \cap T_\alpha| \leq 1$, and if $\beta < \alpha$, then by maximality $b \cap T_\alpha \neq \emptyset$ implies that $b \cap T_\beta \neq \emptyset$. We say that b is unbounded (or cofinal) if for all $\alpha < ht(T)$, $b \cap T_\alpha \neq \emptyset$.

Definition 19. *The* **tree property** *holds at* κ*, for a regular cardinal* κ*, if every tree of height* κ *and levels of size less than* κ*, has an unbounded branch. We denote this by* TP_κ*.*

Below we list some facts about the tree property:

(1) (König) TP holds at ω.
(2) (Aronszajn) TP fails at ω_1.
(3) TP can hold at ω_2 (and above), assuming some large cardinals.
(4) $\square^*_\kappa$ implies that the tree property fails at κ^+.

Definition 20. *An inaccessible cardinal* κ *is weakly compact if it satisfies the tree property.*

There are several equivalent definitions of a weakly compact. We list two of them for completeness:

(1) κ is weakly compact iff κ is inaccessible and $\mathcal{L}_{\kappa,\omega}$ satisfies the Weak Compactness Theorem.
Here the language $\mathcal{L}_{\kappa,\omega}$ contains conjunctions and disjunctions of size less than κ. It satisfies the weak Compactness Theorem if for every set of sentences $\Sigma \subset \mathcal{L}_{\kappa,\omega}$ with $|\Sigma| \leq \kappa$, if every $S \subset \Sigma$ with $|S| < \kappa$ has a model, then Σ has a model.
(2) κ is weakly compact iff every function $F : [\kappa]^2 \to 2$, there is a set $H \subset \kappa$ of size κ such that F is constant on H. Such a set is called *homogeneous.*

It turns out that TP at ω_2 is equiconsistent with the existence of a weakly compact cardinal. More precisely:

Theorem 21. *(Mitchell) If* κ *is weakly compact, then there is a forcing extension, in which* κ *is* $\aleph_2$*, and the tree property holds at* $\aleph_2$*.*

Theorem 22. *(Silver)* $\aleph_2$ *has the tree property, then* $\aleph_2$ *is weakly compact in* L*.*

Later in the course we will go over the proof of Mitchell's theorem. hat measurable cardinals are weakly compact.